point A_2. Let us increase $|\alpha|$. At some α, $z_k = z_2$. Further increase in $|\alpha|$ leads to a value $\alpha = \alpha_3(\beta, z_1, z_2)$ such that the curve G joins the points A_1 and A_2. At the same time, f(z) has one zero in the interval $[z_1; z_2]$. We then find the value $\alpha = \alpha_4(\beta, z_1, z_2)$ with the next largest modulus for which two zeros of the function f(z) lie within the interval. Continuing this process, we find the spectrum of stable gravitational—acoustic oscillations.

We now consider the interval $0 < \alpha < \beta$. It can be shown that for these values of α the function G increases monotonically and $G \to 1$ as $z \to +\infty$. Therefore, on this interval there are no values of α satisfying the imposed conditions.

Consider $\alpha > \beta$. The function G decreases monotonically with increasing z and tends to 1 as $z \to +\infty$. Therefore, for all $\alpha > \beta$ the curve G passes below the point A_2. Thus, we have shown that there exists only the one unstable mode (15).

I should like to thank S. I. Anisimov for constant interest in the work.

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GAS-KINETIC THEORY OF LUBRICATION

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An equation of the gas-kinetic theory of lubrication is obtained under the assumption of incompressibility of the gas on the basis of solution of the Boltzmann equation by the moment method with a special approximating function. In the limit of a small Knudsen number calculated using the minimal gap, the equation goes over into Reynolds's well-known equation. Reynolds's problem of a lubricating layer of gas between two closely spaced planes is considered. In the limit of a small Knudsen number, agreement with the well-known solution of the hydrodynamic theory is obtained. A comparison is made with the solution obtained by the hydrodynamic method with slip boundary conditions under neglect of the compressibility of the gas.

The basic equation of the hydrodynamic theory of lubrication is Reynolds's equation [1]. This equation describes the motion of a liquid in the gap between lubricated surfaces.

The development of the modern technique of gas lubrication poses the problem of developing a theory of gas bearings with ultrathin gaps equal to or less than the mean free path λ of the gas molecules [2]. At the same time, the Knudsen number K can take arbitrary values. In particular, in devices with a high density of recording computer information the gap between the head and the carrier of the information may be fractions of a micrometer [3]. The surfaces are worked to a high degree. According to the data of [4], the microrelief of the head is less than 0.01 μm, an order of magnitude less than the molecule mean free path. When ordinary gas bearings begin to move and stop the minimal gap is also comparable with the mean free path [2].

To describe gas lubrication with gaps corresponding to small Knudsen numbers, one uses a modified Reynolds's equation obtained using slip boundary conditions in the hydrodynamic equations [3-6]. Experimental data show that there are conditions under which this equation is invalid. According to the data of [3], the discrepancy between experiment and theory is in the range from 20 to 40% for variation of the Knudsen number from 0.1 to 0.173. In the region of moderate and large Knudsen numbers, there is no

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justification for using the modified Reynolds's equation.

It would be worth using the methods of the kinetic theory of gases to construct a theory of gas bearings with ultrathin gaps. In [5, 7] a representation of the function that approximates the distribution function in the moment method of rarefied gas dynamics for gas lubrication problems was considered. This representation was used in [5, 8, 9] to study some problems.

1. We consider the isothermal problem of the relative motion of two surfaces with velocities $\mathbf{V}_1$ and $\mathbf{V}_2$, the space between them being filled with gas. We assume V_1, $V_2 \ll 4g$. Here, $4g = \sqrt{2kT/\pi m}$ is the mean thermal velocity of the gas molecules, k is Boltzmann's constant, and T is the temperature of the gas. The gap between the surfaces is much less than the characteristic linear dimension of each surface.

We introduce a Cartesian coordinate system xyz. The vector $\mathbf{V}_1$ lies in the plane yz. We place the coordinate origin on the surface moving with velocity $\mathbf{V}_1$. The plane xy coincides with the tangent plane to this surface. We specify the surface by the equation $z = z_1(x, y)$. The second surface is specified by the equation $z = z_2(x, y) = h(x, y) + z_1(x, y)$. Here, $h(x, y)$ is the gap between the surfaces. We bound the region of consideration by a cylindrical surface $\varphi(x, y) = 0$.

We require

$$\left(\frac{\partial z_1}{\partial x}\right)^2+\left(\frac{\partial z_1}{\partial y}\right)^2 \ll 1; \qquad \left(\frac{\partial z_2}{\partial x}\right)^2+\left(\frac{\partial z_2}{\partial y}\right)^2 \ll 1 \tag{1.1}$$

$$|z_1(x, y)| \ll |h(x, y)| \tag{1.2}$$

We describe the motion of the rarefied gas in the film by Boltzmann's equation [10]

$$\nabla(\mathbf{c}nf)=\Delta_c nf \tag{1.3}$$

Here, c is the vector of the molecular velocity, n is the molecule number density, $\Delta_c nf$ is the Boltzmann collision integral, and nf is the velocity distribution function of the molecules.

We take a diffuse law of reflection [10] of the gas molecules from the bounding surfaces, and outside the considered region we assume the pressure is given:

$$\begin{gathered} nf(c)=n_{w1}f_0(\mathbf{c}, T, \mathbf{V}_1), \quad z=z_1(x, y), \quad c_z>0, \quad \mathbf{V}_1=\{U_1, 0, W_1\} \\ nf(c)=n_{w2}f_0(\mathbf{c}, T, \mathbf{V}_2), \quad z=z_2(x, y), \quad c_z<0, \quad \mathbf{V}_2=\{U_2, V, W_2\} \\ n=n_0, \quad \varphi(x, y)=0 \end{gathered} \tag{1.4}$$

Here, $nf_0(\mathbf{c}, T, \mathbf{V})$ is the Maxwell distribution function, and n_{w1} and n_{w2} are number densities satisfying no-flow conditions on the surfaces. At the same time, by virtue of (1.1)

$$\mathbf{c}\mathbf{n}_{1,2} = c_x\frac{\partial z_{1,2}}{\partial x} + c_y\frac{\partial z_{1,2}}{\partial y} + c_z \approx c_z$$

where $\mathbf{n}_{1,2}$ are the vectors of the normal to the respective surfaces.

To solve Boltzmann's equation (1.3) with the boundary conditions (1.4), we use the moment method, in which the distribution function is approximated by a function which depends on a finite number of parameters. The parameters that determine the distribution function are found by solving a corresponding number of transport equations. We use the approximating function constructed in [5, 7] for problems of the flow of isothermal gas in thin films:

$$nf(c)=n_0f_{00}\left[1+\nu(\mathbf{r})+\frac{c_x}{RT}u_{1,2}(\mathbf{r})+\frac{c_y}{RT}v_{1,2}(\mathbf{r})+\frac{c_z}{RT}w(\mathbf{r})\right], \qquad f_{00}=(2\pi RT)^{-3/2}\exp\left(-\frac{c^2}{2RT}\right) \tag{1.5}$$

where n_0 = const, $|\nu(\mathbf{r})| \ll 1$, r is the radius vector of a point of space, R is the universal gas constant, and the subscripts 1 and 2 correspond to $c_z > 0$ and $c_z < 0$.

To determine the six unknown functions $\nu(\mathbf{r})$, $u_{1,2}(\mathbf{r})$, $v_{1,2}(\mathbf{r})$, $w(\mathbf{r})$, we choose the six transport equations obtained by multiplying Boltzmann's equation by 1, c_x, c_y, c_z, c_xc_z, c_yc_z

and integrating over the complete velocity space of the molecules under the assumption of Maxwellian molecules.

The approximating function (1.5) has the property that the continuity equation is identical to the energy conservation equation [5, 7].

Using the approximating function (1.5), we calculate the moments in the system of transport equations. For example,

$$n\langle 1\rangle=n_0(1+\nu);\quad n\langle c_x\rangle=n_0u_+/2;\quad n\langle c_x^2\rangle=2\pi g^2n_0(1+\nu),\quad u_\pm=u_1\pm u_2;\quad v_\pm=v_1\pm v_2$$

We introduce dimensionless variables: $u_\pm'=u_\pm/g$; $v_\pm'=v_\pm/g$; $w'=w/g$; $\nu'=2\pi\nu$.

Substituting the moments of the approximating function in the system of transport equations, we obtain

$$\frac{\partial u_+}{\partial x}+\frac{\partial v_+}{\partial y}+2\frac{\partial w}{\partial z}=0,\quad \frac{\partial \nu}{\partial x}+\frac{\partial u_-}{\partial z}=0,\quad \frac{\partial \nu}{\partial y}+\frac{\partial v_-}{\partial z}=0$$
$$\frac{\partial u_-}{\partial x}+\frac{\partial v_-}{\partial y}+\frac{\partial \nu}{\partial z}=0,\quad 2\frac{\partial w}{\partial x}+\frac{\partial u_+}{\partial z}=-\frac{u_-}{\lambda},\quad 2\frac{\partial w}{\partial y}+\frac{\partial v_+}{\partial z}=-\frac{v_-}{\lambda} \tag{1.6}$$

From (1.4), using (1.5), we obtain the boundary conditions for this system:

$$z=z_1(x,y),\quad u_1=U_1/g=U_1',\quad v_1=0,\quad w=W_1/g=W_1'$$
$$z=z_2(x,y),\quad u_2=U_2/g=U_2',\quad v_2=V/g=V',\quad w=W_2/g=W_2',\quad \varphi(x,y)=0,\quad \nu=0 \tag{1.7}$$

Using the last five equations of the system to express the derivatives of the concentration with respect to the coordinates, and taking into account the geometry of the problem, we can show that

$$\left|\frac{\partial \nu}{\partial z}\right| \ll \left|\frac{\partial \nu}{\partial x}\right| + \left|\frac{\partial \nu}{\partial y}\right|,\quad \frac{\partial \nu}{\partial z}=0 \tag{1.8}$$

In addition, estimating by means of the first equation of the system (1.6) the quantities u_+, v_+, w, we can readily show that in the last two equations of the system (1.6) it is possible to ignore $\partial w/\partial x$ compared with $\partial u_+/\partial z$ and also $\partial w/\partial y$ compared with $\partial v_+/\partial z$.

Then, using also (1.8), we integrate the system (1.6):

$$u_-=-\frac{\partial \nu}{\partial x}z+C_1(x,y),\quad v_-=-\frac{\partial \nu}{\partial y}z+C_2(x,y),\quad u_+=\frac{\partial \nu}{\partial x}\frac{z^2}{2\lambda}-\frac{z}{\lambda}C_1(x,y)+C_3(x,y)$$

$$v_+=\frac{\partial \nu}{\partial y}\frac{z^2}{2\lambda}-\frac{z}{\lambda}C_2(x,y)+C_4(x,y)\qquad \frac{z^2}{2\lambda}\frac{\partial^2 \nu}{\partial x^2}-\frac{z}{\lambda}\frac{\partial C_1}{\partial x}+\frac{\partial C_3}{\partial x}+\frac{z^2}{2\lambda}\frac{\partial^2 \nu}{\partial y^2}-\frac{z}{\lambda}\frac{\partial C_2}{\partial y}+\frac{\partial C_4}{\partial y}+2\frac{\partial w}{\partial z}=0$$

We integrate the last equation over z from $z_1(x, y)$ to $z_2(x, y)$. Using the boundary conditions (1.7) for w, we obtain

$$\frac{z_2^3-z_1^3}{6\lambda}\Delta\nu-\frac{z_2^2-z_1^2}{2\lambda}\left(\frac{\partial C_1}{\partial x}+\frac{\partial C_2}{\partial y}\right)+h\left(\frac{\partial C_3}{\partial x}+\frac{\partial C_4}{\partial y}\right)+2\frac{W_2-W_1}{g}=0,\quad \Delta\nu=\frac{\partial^2\nu}{\partial x^2}+\frac{\partial^2\nu}{\partial y^2} \tag{1.9}$$

From the boundary conditions (1.7), we determine the constants of integration C_1, C_2, C_3, C_4. Substituting their values in (1.9) and using (1.2), we arrive at an equation for determining the concentration:

$$\frac{h^2\eta}{3}\Delta\nu-\frac{h}{2}(\eta+1)\left[\frac{\partial}{\partial x}\left(h\frac{\partial \nu}{\partial x}\right)+\frac{\partial}{\partial y}\left(h\frac{\partial \nu}{\partial y}\right)\right]=\frac{V'}{1+\eta^{-1}}\frac{\partial h}{\partial y}+\frac{U_2'-U_1'}{1+\eta^{-1}}\frac{\partial h}{\partial x}+2(W_1'-W_2'),\quad \eta=\frac{h}{2\lambda} \tag{1.10}$$

This equation is the equation of the gas-kinetic theory of lubrication without allowance for the compressibility of the gas.

We estimate the velocities of the surfaces for which the gas lubrication can be assumed to be incompressible. We take $h\sim h_0$, $x\sim a$, $y\sim b$. To be specific, we take

$a < b$. Then the operator on the left-hand side of (1.10) has the order

$$-\frac{\nu h_0^2}{2a^2}\left(\frac{\eta_0}{3}+1\right), \quad \eta_0=\frac{h_0}{2\lambda}$$

Taking into account only the component W_2' of the velocity of the surface $z = z_2(x, y)$, we find that the condition $|\nu| \ll 1$ will be satisfied when

$$|W_2'| \ll \frac{h_0^2}{4a^2}\left(1+\frac{\eta_0}{3}\right)$$

Similarly, for the component U_2' of this surface the condition of incompressibility of the gas will be satisfied when

$$U_2' \ll \frac{h_0}{2a}\left(\frac{4}{3}+\eta_0^{-1}+\frac{\eta_0}{3}\right)$$

We express (1.10) in terms of the pressure and reduce it to a form analogous to the one in [5, 6]. The pressure p is determined by

$$p=2\pi g^2 m n_0+g^2 m \bar{n}_0 \nu.$$

Then from (1.10) we obtain

$$\frac{\partial}{\partial x}\left[h^3(1+3\eta^{-1})\frac{\partial p}{\partial x}\right]+\frac{\partial}{\partial y}\left[h^3(1+3\eta^{-1})\frac{\partial p}{\partial y}\right]-3h^2\eta^{-1}\left[\frac{\partial p}{\partial x}\frac{\partial h}{\partial x}+\frac{\partial p}{\partial y}\frac{\partial h}{\partial y}\right]=$$

$$6\mu\left[2(W_2-W_1)+\frac{U_1-U_2}{1+\eta^{-1}}\frac{\partial h}{\partial x}-\frac{V}{1+\eta^{-1}}\frac{\partial h}{\partial y}\right] \tag{1.11}$$

Here, we have introduced the gas viscosity $\mu = 2\lambda g m n_0$ in accordance with [11].

In the continuum regime, $\lambda/h \to 0$, Eq. (1.11) goes over into Reynolds's well-known equation [1]. In contrast to the modified Reynolds's equation for an incompressible gas, Eq. (1.11) has an additional term that depends on the mean free path, and it also contains a dependence on the mean free path on the right-hand side. By virtue of the approximating function employed, Eq. (1.11) will give the correct solution in the free-molecule limit.

2. On the basis of Eq. (1.11), we consider the Reynolds's problem of a flat bearing. In formulating the problem, we follow [12]. The flat base xy moves with speed U in its plane. The thickness of the gap is determined by $h = h_1 - \alpha x$, where α is the angle of inclination of the bearing, and h_1 is the maximal gap. We denote by h_2 the minimal gap, by $K = \lambda/h_2$ the Knudsen number, and by $H = h/h_2$ the dimensionless thickness of the gap.

In the considered case, Eq. (1.11) and the boundary conditions have the form

$$\frac{d^2p}{dH^2}+\frac{3(H+2K)}{H(H+6K)}\frac{dp}{dH}=\frac{p^\circ}{H(H+2K)(H+6K)}, \quad p^\circ=\frac{6\mu U}{\alpha h_2} \tag{2.1}$$

$$p=p_0, \quad H=H_1=\frac{h_1}{h_2}; \quad p=p_0, \quad H=1 \tag{2.2}$$

Integrating (2.1) with the boundary conditions (2.2), we determine

$$\frac{p-p_0}{p^\circ}=I(H)-\int_1^H\frac{dt}{t(t+6K)^2}\,I(H_1)\Big/\int_1^{H_1}\frac{dt}{t(t+6K)^2}, \quad I(H)=\int_1^H\frac{dt}{(t+6K)^2}+4K\int_1^H\frac{\ln(t+2K)}{t(t+6K)^2}dt \tag{2.3}$$

The force acting on unit length of the bearing is

$$F=\int_0^l(p-p_0)\,dx=\frac{h_2}{\alpha}\int_1^{H_1}(p-p_0)\,dH$$

Here, l is the length of the bearing.

Integrating (2.3), we obtain

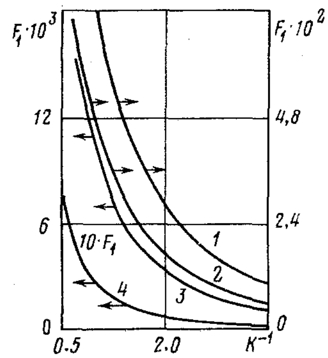

Fig. 1

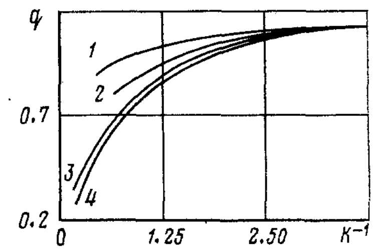

Fig. 2

$$F_1=\frac{F}{F_0}=\int\limits_1^{H_1}\frac{(H_1-H)\,dH}{(H+6K)^2}+4K\int\limits_1^{H_1}\frac{(H_1-H)\ln(H+2K)}{H(H+6K)^2}\,dH-$$
$$\int\limits_1^{H_1}\frac{(H_1-H)\,dH}{H(H+6K)^2}I(H_1)\Bigg/\int\limits_1^{H_1}\frac{dH}{H(H+6K)^2};\qquad F_0=\frac{p^\circ h_2}{\alpha}=\frac{6\mu U}{\alpha^2} \tag{2.4}$$

In the limit $K\to 0$, (2.3) and (2.4) give the well-known solution in the continuum regime [12].

In the free-molecule regime, $K\to\infty$, $K/H_1\to\infty$, we obtain for the force

$$F_1=\frac{(H_1-1)^2(\ln^{-1}H_1-1)\ln 2K}{27K^2}$$

Hydrodynamic theory with slip under neglect of the compressibility of the gas gives the expression

$$F_2=\frac{F}{F_0}=\int\limits_1^{H_1}\frac{(H_1-1)\,dH}{H(H+2K)}-\int\limits_1^{H_1}\frac{dH}{H(H+2K)}\int\limits_1^{H_1}\frac{(H-1)\,dH}{H^2(H+2K)}\Bigg/\int\limits_1^{H_1}\frac{dH}{H^2(H+2K)} \tag{2.5}$$

The considered problem contains the three independent linear parameters λ, h_1, h_2. The solutions (2.4) and (2.5) depend on their two dimensionless combinations $s = 2\lambda/(h_1 - h_2)$ and $K = s(H_1 - 1)/2$.

A computer calculation based on the expressions (2.4) and (2.5) was made. The integrals were calculated by Simpson's formula.* The results of the calculation in accordance with the expression (2.4) are shown in Fig. 1. Curves 1-4 correspond to the values s = 0.8, 1.0, 2.0, and 8.0. The dependence of the ratio $q = F_1/F_2$ of the gas-kinetic solution (2.4) to the hydrodynamic solution (2.5) with slip on the Knudsen number K and the parameter s, which characterizes the slope of the bearing, is shown in Fig. 2 [1) s = 0.8, 2) s = 2.0, 3) s = 8.0, 4) s = 32.0]. As the Knudsen number is increased, the solutions deviate significantly from each other, the difference increasing with decreasing angle of inclination of the bearing (increasing s) and occurs at smaller K.

There are experiments [3] with bearings of finite width for which the calculation must be made by solving the more complicated two-dimensional problem for an equation analogous to (1.11) but obtained with allowance for the compressibility of the gas. At the same time, it should be noted that in the experiments of [3] there is a qualitatively similar behavior of the experimental curve with respect to the hydrodynamic solution with slip. These experiments were made at small angles of inclination of the bearing determined by appreciable lubrication velocities. The analogous experiments at significantly larger angles of inclination [4] give good agreement with the solution on the basis of the hydrodynamic formulation with slip boundary conditions.

The behavior of the gas-kinetic solution relative to the hydrodynamic solution with slip in Reynolds's model problem, which has an analogy with the behavior of the experimental curves with respect to the calculated curves, gives hope that further

*The calculation was made by V. A. Chekalova.

development of the gas-kinetic theory of lubrication will lead to successful solution of problems of calculating real gas bearings with ultrathin gaps.

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